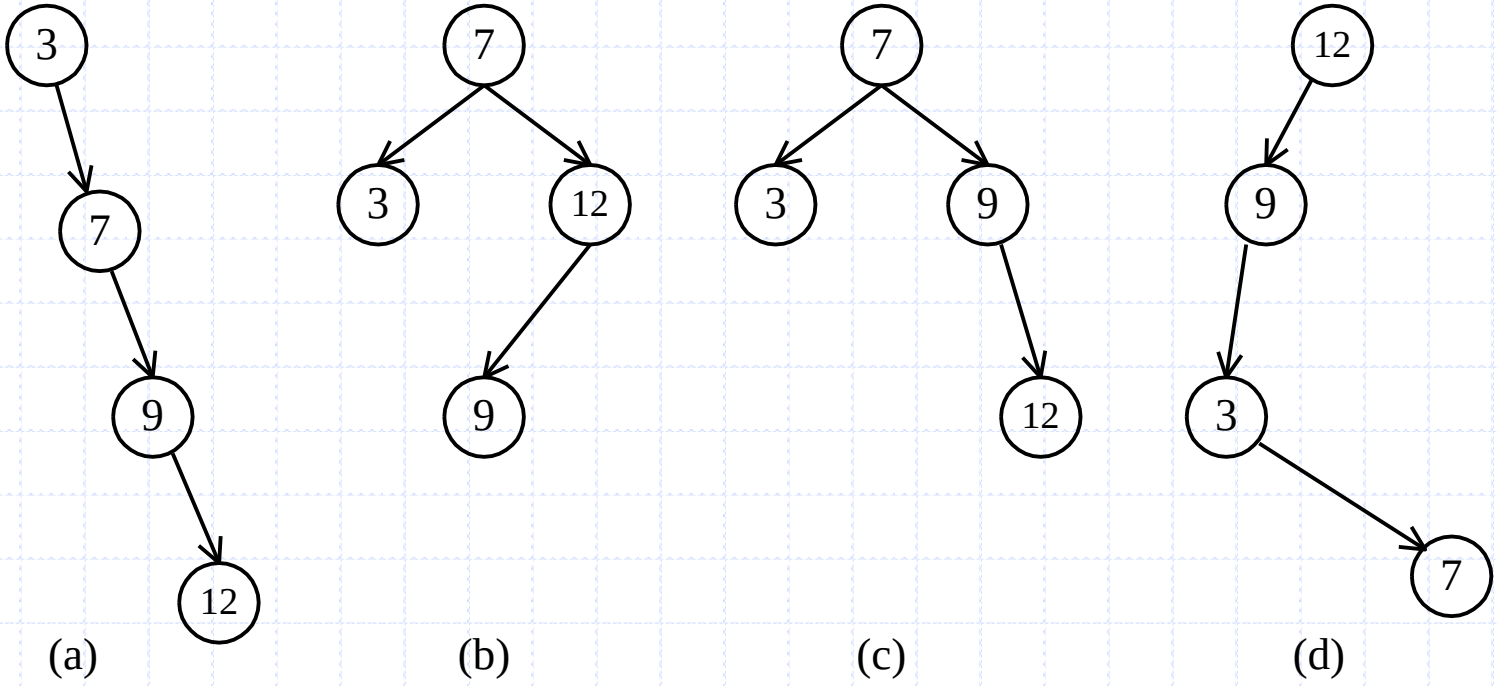
Optimal Binary Search tree

Optimal binary search trees

- e.g. binary search trees for 3, 7, 9, 12;



Optimal binary search trees

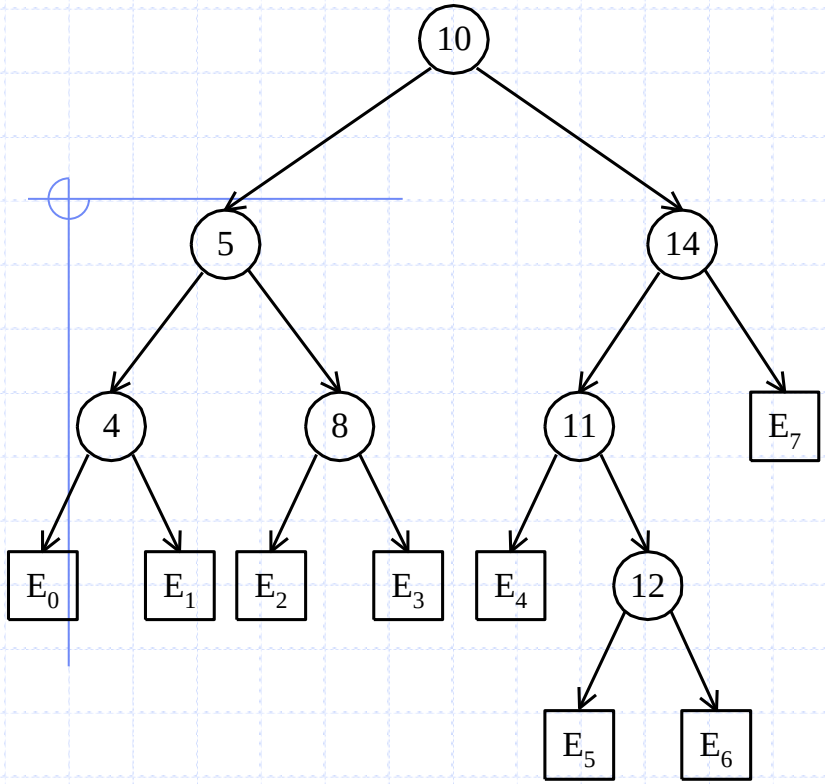
- n identifiers : $a_1 < a_2 < a_3 < \ldots < a_n$

P_i, $1 \le i \le n$: the probability that a_i is searched.

Q_i, $0 \le i \le n$: the probability that x is searched

where $a_i < x < a_{i+1}$ ($a_0 = -\infty$, $a_{n+1} = \infty$).

$$\sum_{i=1}^{n} P_i + \sum_{i=1}^{n} Q_i = 1$$



- Identifiers : 4, 5, 8, 10, 11, 12, 14
- Internal node : successful search, P_i
- External node : unsuccessful search, Q_i

- The expected cost of a binary tree:

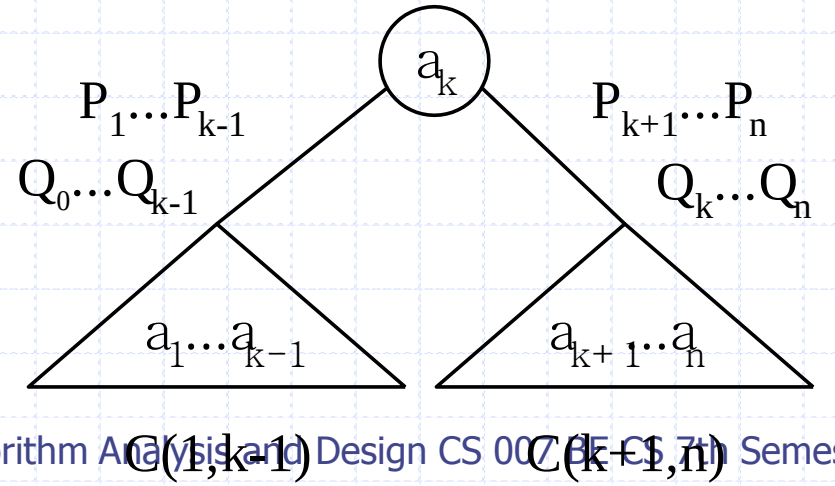
$$\sum_{n=1}^{n} P_i * \text{level}(a_i) + \sum_{n=0}^{n} Q_i * (\text{level}(E_i) - 1)$$

- The level of the root : 1

The dynamic programming approach

- Let C(i, j) denote the cost of an optimal binary search tree containing $a_i,\ldots,a_j$.
- The cost of the optimal binary search tree with a_k as its root :

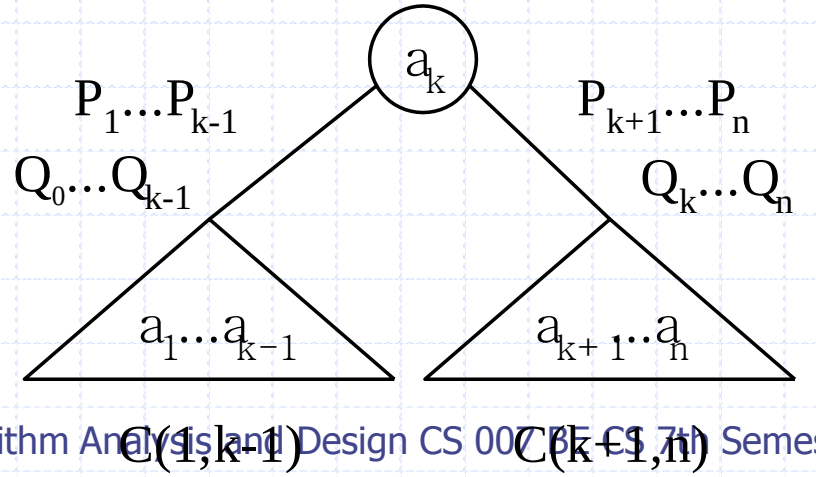
$$C(1,n)=\min_{1\le k\le n}\left\{P_k+\left[Q_0+\sum_{i=1}^{k-1}(P_i+Q_i)+C(1,k-1)\right]+\left[Q_k+\sum_{i=k+1}^{n}(P_i+Q_i)+C(k+1,n)\right]\right\}$$



General formula

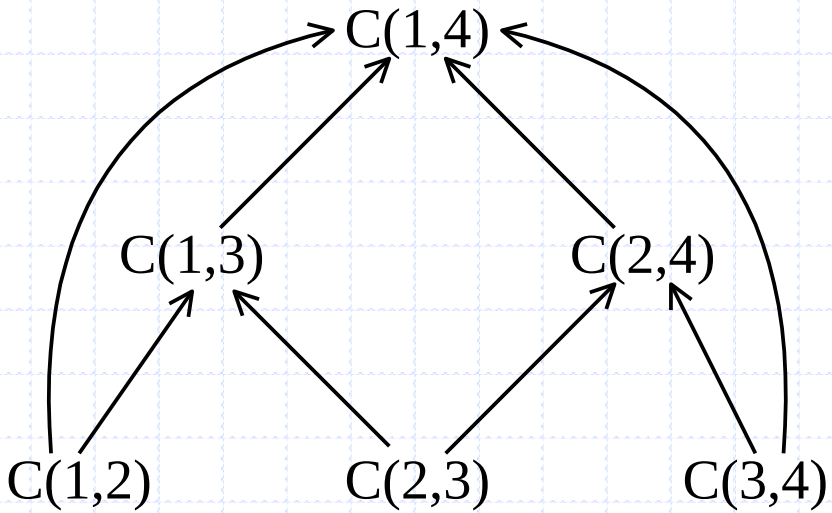
$$C(i,j) = \min_{i \le k \le j} \left\{ P_k + \left[Q_{i-1} + \sum_{m=i}^{k-1} (P_m + Q_m) + C(i, k-1) \right] + \left[Q_k + \sum_{m=k+1}^{j} (P_m + Q_m) + C(k+1, j) \right] \right\}$$

$$= \min_{i \le k \le j} \left\{ C(i, k-1) + C(k+1, j) + Q_{i-1} + \sum_{m=i}^{j} (P_m + Q_m) \right\}$$



Computation relationships of subtrees

- e.g. n=4



- Time complexity : $O(n^3)$
  when j-i=m, there are (n-m) C(i, j)'s to compute.
  Each C(i, j) with j-i=m can be computed in O(m) time.

$$O(\sum_{1\le m\le n} m(n-m)) = O(n^3)$$